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Joint universality of periodic zeta-functions with multiplicative coefficients. II

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Abstract. In the paper, a joint discrete universality theorem for periodic zeta-functions with multiplicative coefficients on the approximation of analytic functions by shifts involving the sequence $\{\gamma_k\}$ of imaginary parts of nontrivial zeros of the Riemann zeta-function is obtained. For its proof, a weak form of the Montgomery pair correlation conjecture is used. The paper is a continuation of [A. Laurinčikas, M. Tekorė, Joint universality of periodic zeta-functions with multiplicative coefficients, *Nonlinear Anal. Model. Control*, 25(5):860–883, 2020] using nonlinear shifts for approximation of analytic functions.

Keywords: joint universality, nontrivial zeros of the Riemann zeta-function, periodic zeta-function, space of analytic functions, weak convergence.

1 Introduction

It is well known that some zeta- and L-functions, and even some classes of Dirichlet series, for example, the Selberg-Steuding class, see [29, 32], are universal in the Voronin sense, i.e., a wide class of analytic functions can be approximated by one and the same zeta-function. For example, in the case of the Riemann zeta-function $\zeta(s)$, $s = \sigma + \mathrm{i}t$, analytic nonvanishing functions on the strip $D = \{s \in \mathbb{C}\colon 1/2 < \sigma < 1\}$ are approximated by shifts $\zeta(s + \mathrm{i}\tau)$, $\tau \in \mathbb{R}$ (continuous case), or shifts $\zeta(s + \mathrm{i}kh)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $h > 0$ (discrete case); see [1, 6, 13, 24, 32].

The above shifts are very simple, τ and kh occur in them linearly. It turned out that the approximation remains valid also with more general shifts. A significant progress in this direction was made by Pańkowski [31] using the shifts $\zeta(s+\mathrm{i}\varphi(\tau))$ and $\zeta(s+\mathrm{i}\varphi(k))$

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with $\varphi(\tau) = \tau^{\alpha} \log^{\beta} \tau$ and a wide class of reals α and β. The papers [22] and [35] are also devoted to approximation of analytic functions by generalized shifts of zeta-functions. In [5], the shifts $\zeta(s + \mathrm{i}h\gamma_k)$ were applied, where $\{\gamma_k\colon k \in \mathbb{N}\} = \{\gamma_k\colon 0 < \gamma_1 < \cdots \leqslant \gamma_k \leqslant \gamma_{k+1} \leqslant \cdots\}$ is the sequence of imaginary parts of nontrivial zeros of the Riemann zeta-function.

Universality in the Voronin sense also has its joint version. In the joint case, a collection of analytic functions is approximated simultaneously by a collection of shifts of zeta- or L-functions. The first joint universality theorem belongs to Voronin who proved [36] the joint universality of Dirichlet L-functions $L(s, \chi_j)$, $j = 1, \dots, r$. Obviously, in joint universality theorems, the approximating shifts must be in some sense independent. Voronin required [36] for this the pairwise nonequivalence of Dirichlet characters, i.e., in fact, he considered joint universality of different Dirichlet L-functions. On the other hand, as it was observed by Pańkowski [31], the independence of approximating shifts of Dirichlet L-functions can be ensured by different functions $\varphi_j(\tau)$ in shifts $L(s + \mathrm{i}\varphi_j(\tau), \chi_j)$ or $L(s + \mathrm{i}\varphi_j(k), \chi_j)$ even with the same characters χ_j. This observation extends significantly classes of jointly universal functions. For example, the joint universality with generalized shifts was obtained in [16] and [20].

In general, joint universality of zeta-functions was widely studied, and many results are known; see, for example, general results obtained in [7–11, 14, 26, 30] and other papers by authors of the mentioned works. In this note, we focus on joint universality of so-called periodic zeta-functions with generalized shifts involving the sequence $\{\gamma_k\colon k \in \mathbb{N}\}$ of imaginary parts of nontrivial zeros of the function $\zeta(s)$. We will mention some joint universality results involving the latter sequence. Note that the behaviour of the sequence $\{\gamma_k\}$, as of nontrivial zeros of $\zeta(s)$, is very complicated, and at the moment, its known properties are not sufficient for the proof of universality. Therefore, in [5], the conjecture that, for $c > 0$,

$$\sum_{\substack{\gamma_k, \gamma_l \leqslant T \\ |\gamma_k - \gamma_l| < c/\log T}} 1 \ll T \log T \tag{1}$$

was introduced. This conjecture is inspired by the Montgomery pair correlation conjecture [28] that

$$\sum_{\substack{\gamma_k, \gamma_l \leqslant T \\ 2\pi\alpha_1/\log T \leqslant \gamma_k - \gamma_l \leqslant 2\pi\alpha_2/\log T}} 1 \sim \left(\int_{\alpha_1}^{\alpha_2} \left(1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 \right) \mathrm{d}u + \delta(\alpha_1, \alpha_2) \right) \frac{T}{2\pi} \log T,$$

where $\alpha_1 < \alpha_2$ are arbitrary real numbers, and

$$\delta(\alpha_1, \alpha_2) = \begin{cases} 1 & \text{if } 0 \in [\alpha_1, \alpha_2], \\ 0 & \text{otherwise.} \end{cases}$$

Now we will state a joint universality theorem for Dirichlet L-functions involving the sequence $\{\gamma_k\}$ obtained in [18]. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous nonvanishing functions on K that are analytic in the interior of K.

Theorem 1. *Suppose that $\chi_1, \dots, \chi_r$ are pairwise nonequivalent Dirichlet characters, and estimate* (1) *is true. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \Bigl\{ 1 \leqslant k \leqslant N\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \bigl| L(s + \mathrm{i}h\gamma_k, \chi_j) - f_j(s) \bigr| < \varepsilon \Bigr\} > 0.$$

Moreover "$\liminf$" *can be replaced by* "$\lim$" *for all but at most countably many $\varepsilon > 0$.*

Here $\#A$ denotes the cardinality of the set A, and N runs over the set $\mathbb{N}$.

Now we recall the definition of the periodic zeta-function, which is an object of investigation of the present note. Let $\mathfrak{a} = \{a_m\colon m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. Then the periodic zeta-function $\zeta(s; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}$$

and has an analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\frac{1}{q} \sum_{l=1}^{q} a_l.$$

The sequence $\mathfrak{a}$ is called multiplicative if $a_1 = 1$ and $a_{mn} = a_m a_n$ for all coprimes $m, n \in \mathbb{N}$. If $0 < \alpha \leqslant 1$ is a fixed number, then the function

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}, \quad \sigma > 1,$$

and its meromorphic continuation are called the periodic Hurwitz zeta-function. In [15] and [3], under hypothesis (1), joint universality theorems involving sequence $\{\gamma_k\}$ for the pair consisting from the Riemann and Hurwitz zeta-functions and their periodic analogues, respectively, were obtained, while in [23], such theorems were proved for Hurwitz zeta-functions.

For $j = 1, \dots, r$, let $\mathfrak{a}_j = \{a_{jm}\colon m \in \mathbb{N}\}$ be a periodic sequences of complex numbers with minimal period $q_j \in \mathbb{N}$, and let $\zeta(s; \mathfrak{a}_j)$ be the corresponding zeta-function. The main result of the paper is the following theorem.

Theorem 2. *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative, $h_1, \dots, h_r$ are positive algebraic numbers linearly independent over the field of rational numbers, and estimate* (1) *is true. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \Bigl\{ 1 \leqslant k \leqslant N\colon \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \bigl| \zeta(s + \mathrm{i}h_j\gamma_k; \mathfrak{a}_j) - f_j(s) \bigr| < \varepsilon \Bigr\} > 0.$$

Moreover "$\liminf$" *can be replaced by* "$\lim$" *for all but at most countably many $\varepsilon > 0$.*

In [21], joint continuous universality theorems for periodic zeta-functions with shifts defined by means of certain differentiable functions were obtained.

2 The sequence $\{\gamma_k\}$

From the functional equation for the Riemann zeta-function

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

it follows that $\zeta(-2m) = 0$ for all $m \in \mathbb{N}$, and the zeros $s = -2m$ of $\zeta(s)$ are called trivial. Moreover, it is known that $\zeta(s)$ has infinitely many of so-called complex nontrivial zeros $\rho_k = \beta_k + \mathrm{i}\gamma_k$ lying in the strip $\{s \in \mathbb{C}\colon 0 < \sigma < 1\}$. The famous Riemann hypothesis, one of seven Millennium problems, asserts that $\beta_k = 1/2$, i.e., all nontrivial zeros lie on the critical line $\sigma = 1/2$. There exists a conjecture that all nontrivial zeros of $\zeta(s)$ are simple.

We recall some properties of the sequence

$$\{\gamma_k\colon k \in \mathbb{N}\} = \{\gamma_k\colon 0 < \gamma_1 < \cdots \leqslant \gamma_k \leqslant \gamma_{k+1} \leqslant \cdots\}.$$

By the definition, a sequence $\{x_k\colon k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1, if, for every subinterval $(a, b] \subset (0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I_{(a,b]}\big(\{x_k\}\big) = b - a,$$

where $I_{(a,b]}$ is the indicator function of $(a, b]$, and $\{u\}$ denotes the fractional part of $u \in \mathbb{R}$. Though the sequence $\{\gamma_k\}$ is distributed irregularly, the following statement is true for it.

Lemma 1. *The sequence $\{\gamma_k a\colon k \in \mathbb{N}\}$ with every $a \in \mathbb{R} \setminus \{0\}$ is uniformly distributed modulo* 1.

Proof. Proof of the lemma is given in [33], and in the above form, was applied in [5]. □

For convenience, we recall the Weyl criterion on the uniform distribution modulo 1; see, for example, [12].

Lemma 2. *A sequence $\{x_k\colon k \in \mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo* 1 *if and only if, for every $m \in \mathbb{Z} \setminus \{0\}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{e}^{2\pi \mathrm{i} m x_k} = 0.$$

Obviously, the uniform distribution modulo 1 of the sequence shows its nonlinear character.

The following statement is well known; see, for example, [34].

Lemma 3. *For $k \to \infty$,*

$$\gamma_k = \frac{2\pi k}{\log k}\big(1 + o(1)\big).$$

3 Limit theorems

Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. We will derive Theorem 2 from a limit theorem on the weak convergence of probability measures in the space

$$H^r(D) = \underbrace{H(D) \times \cdots \times H(D)}_{r}.$$

Therefore, we start with a certain probability model.

Let $\mathcal{B}(\mathbb{X})$ be the Borel σ-field of the space $\mathbb{X}$, and $\mathbb{P}$ denote the set of all prime numbers. Define

$$\Omega = \prod_{p \in \mathbb{P}} \mathbb{X}_p,$$

where $\mathbb{X}_p = \{s \in \mathbb{C}\colon |s| = 1\}$ for all $p \in \mathbb{P}$. Then Ω is a compact topological Abelian group. Moreover, let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then again Ω^r is a compact topological Abelian group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H^r can be defined. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$. Denote by $\omega(p)$ the pth component, $p \in \mathbb{P}$, of an element $\omega_j \in \Omega_j$, $j = 1, \dots, r$. For brevity, let $\omega = (\omega_1, \dots, \omega_r) \in \Omega^r$, $\omega_1 \in \Omega_1, \dots, \omega_r \in \Omega_r$, $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_r)$, and on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, define the $H^r(D)$-valued random element

$$\underline{\zeta}(s, \omega; \underline{\mathfrak{a}}) = \bigl(\zeta(s, \omega_1; \mathfrak{a}_1), \dots, \zeta(s, \omega_r; \mathfrak{a}_r)\bigr),$$

where

$$\zeta(s, \omega_j; \mathfrak{a}_j) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{a_{jp^l} \omega_j^l(p)}{p^{ls}}\right), \quad j = 1, \dots, r.$$

Note that the latter products, for almost all ω_j, are uniformly convergent on compact subsets of the strip D. Since the periodic sequences $\mathfrak{a}_j$, $j = 1, \dots, r$, are bounded, the proofs of the above assertions completely coincides with those of Lemma 5.1.6 and Theorem 5.1.7 from [13]. More general results are given in [1]. Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s, \omega; \underline{\mathfrak{a}})$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H^r\bigl\{\omega \in \Omega^r\colon \underline{\zeta}(s, \omega; \underline{\mathfrak{a}}) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H^r(D)\bigr).$$

Put $\underline{h} = (h_1, \dots, h_r)$, and, for $A \in \mathcal{B}(H^r(D))$, define

$$P_N(A) = \frac{1}{N} \#\bigl(1 \leqslant k \leqslant N\colon \underline{\zeta}(s + \mathrm{i}\underline{h}\gamma_k; \underline{\mathfrak{a}}) \in A\bigr),$$

where

$$\underline{\zeta}(s; \underline{\mathfrak{a}}) = \bigl(\zeta(s; \mathfrak{a}_1), \dots, \zeta(s; \mathfrak{a}_r)\bigr).$$

In this section, we will prove the following limit theorem.

Theorem 3. *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative, $h_1, \dots, h_r$ are positive algebraic numbers linearly independent over $\mathbb{Q}$, and estimate* (1) *is valid. Then P_N converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.*

We start the proof of Theorem 3, as usual, with a limit lemma in the space Ω^r. In this lemma, the uniform distribution modulo 1 of the sequence $\{\gamma_k a\}$, $a \in \mathbb{R} \setminus \{0\}$, and the property of the numbers $h_1, \dots, h_r$ essentially are applied.

For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_N(A) = \frac{1}{N} \#\bigl\{1 \leqslant k \leqslant N\colon \bigl(\bigl(p^{-\mathrm{i}h_1\gamma_k}\colon p \in \mathbb{P}\bigr), \dots, \bigl(p^{-\mathrm{i}h_r\gamma_k}\colon p \in \mathbb{P}\bigr)\bigr) \in A\bigr\}.$$

Before the statement of a limit theorem for Q_N, we recall one result of Diophantine type.

Lemma 4. *Suppose that $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ are algebraic numbers such that the logarithms $\log \lambda_1, \dots, \log \lambda_r$ are linearly independent over $\mathbb{Q}$. Then, for any algebraic numbers $\beta_0, \dots, \beta_r$, not all zero, we have*

$$|\beta_0 + \beta_1 \log \lambda_1 + \dots + \beta_r \log \lambda_r| > H^{-C},$$

where H is the maximum of the heights of $\beta_0, \beta_1, \dots, \beta_r$, and C is an effectively computable number depending on r and the maximum of the degrees of $\beta_0, \beta_1, \dots, \beta_r$.

The lemma is the well-known Baker theorem on logarithm forms; see, for example [2].

Lemma 5. *Suppose that $h_1, \dots, h_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then Q_N converges weakly to the Haar measure m_H^r as $N \to \infty$.*

Proof. As usual, we apply the Fourier transform method. The characters of the group Ω^r are of the form

$$\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p),$$

where the star "$*$" shows that only a finite number of integers k_{jp} are distinct from zero. Therefore, the Fourier transform of Q_N is

$$g_N(\underline{k}_1, \dots, \underline{k}_r) = \int_{\Omega^r} \Biggl(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p)\Biggr) \mathrm{d}Q_N,$$

where $\underline{k}_j = (k_{jp}\colon k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \dots, r$. Thus, by the definition of Q_N,

$$\begin{aligned} g_N(\underline{k}_1, \dots, \underline{k}_r) &= \frac{1}{N} \sum_{k=1}^{N} \prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} p^{-\mathrm{i}h_j k_{jp} \gamma_k} \\ &= \frac{1}{N} \sum_{k=1}^{N} \exp\Biggl\{-\mathrm{i}\gamma_k \sum_{j=1}^{r} h_j \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p\Biggr\}. \end{aligned} \tag{2}$$

Obviously,

$$g_N(\underline{0}, \ldots, \underline{0}) = 1. \tag{3}$$

Now, suppose that $\underline{k} \neq (\underline{0}, \ldots, \underline{0})$. Then there exists $j \in \{1, \ldots, r\}$ such that $\underline{k}_j \neq \underline{0}$. Thus, there exists a prime number p such that $k_{jp} \neq 0$. Define

$$a_p = \sum_{j=1}^{r} h_j k_{jp}.$$

Then, in view of a property of the numbers $h_1, \ldots, h_r$, we have $a_p \neq 0$. The numbers a_p are algebraic, and the set $\{\log p\colon p \in \mathbb{P}\}$ is linearly independent over $\mathbb{Q}$. Therefore, by Lemma 4,

$$a_{\underline{k}_1, \ldots, \underline{k}_r} \stackrel{\text{def}}{=} \sum_{j=1}^{r} h_j \sideset{}{^*}\sum_{p \in \mathbb{P}} k_{jp} \log p = \sideset{}{^*}\sum_{p \in \mathbb{P}} a_p \log p \neq 0.$$

Hence, in virtue of Lemma 1, the sequence

$$\left\{ \frac{1}{2\pi} \gamma_k a_{\underline{k}_1, \ldots, \underline{k}_r} \colon k \in \mathbb{N} \right\}$$

is uniformly distributed modulo 1. This, together with (2) and Lemma 2, shows that, in the case $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$,

$$\lim_{N \to \infty} g_N(\underline{k}_1, \ldots, \underline{k}_r) = 0.$$

Thus, in view of (3),

$$\lim_{N \to \infty} g_N(\underline{k}_1, \ldots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) = (\underline{0}, \ldots, \underline{0}), \\ 0 & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0}), \end{cases}$$

and the lemma is proved because the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H^r. □

Lemma 5 implies a limit lemma in the space $H^r(D)$ for absolutely convergent Dirichlet series. Let, for a fixed $\theta > 1/2$,

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n} \right)^{\theta} \right\}, \quad m, n \in \mathbb{N},$$

and

$$\zeta_n(s; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm} v_n(m)}{m^s}, \quad j = 1, \ldots, r.$$

Then the latter series are absolutely convergent for $\sigma > 1/2$. Actually, since $v_n(m) \ll m^{-L/n^{\theta}}$ with every $L > 0$, the latter series are absolutely convergent even in the whole complex plane. For $\mathcal{B}(H^r(D))$, define

$$V_{N,n}(A) = \frac{1}{N}\#\bigl\{1 \leqslant k \leqslant N\colon\ \underline{\zeta}_n(s + \mathrm{i}\underline{h}\gamma_k; \underline{\mathfrak{a}}) \in A\bigr\},$$

where

$$\underline{\zeta}_n(s; \underline{\mathfrak{a}}) = \bigl(\zeta_n(s; \mathfrak{a}_1), \dots, \zeta_n(s; \mathfrak{a}_r)\bigr).$$

Moreover, let

$$\zeta_n(s, \omega_j; \mathfrak{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm}\omega_j(m)v_n(m)}{m^s}, \quad j = 1, \dots, r,$$

$$\underline{\zeta}_n(s, \omega; \underline{\mathfrak{a}}) = \bigl(\zeta_n(s, \omega_1; \mathfrak{a}_1), \dots, \zeta_n(s, \omega_r; \mathfrak{a}_r)\bigr),$$

and let $u_n : \Omega^r \to H^r(D)$ be given by the formula

$$u_n(\omega) = \underline{\zeta}_n(s, \omega; \underline{\mathfrak{a}}).$$

Lemma 6. *Suppose that $h_1, \dots, h_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then $V_{N,n}$, as $N \to \infty$, converges weakly to a measure $V_n =^{def} m_H^r u_n^{-1}$, where*

$$m_H^r u_n^{-1}(A) = m_H^r\bigl(u_n^{-1}A\bigr), \quad A \in \mathcal{B}\bigl(H^r(D)\bigr).$$

Proof. Since the series for $\zeta_n(s, \omega_j; \mathfrak{a}_j)$ are absolutely convergent for $\sigma > 1/2$, the function u_n is continuous, hence $(\mathcal{B}(\Omega^r), \mathcal{B}(H^r(D)))$-measurable. Therefore, the measure V_n is defined correctly. The definitions of Q_N, $V_{N,n}$ and u_n imply the equality $V_{N,n} = Q_N u_n^{-1}$. Therefore, the lemma follows from Lemma 5 and a preservation of weak convergence under continuous mappings; see [4, Thm. 5.1]. □

The limit measure V_n in Lemma 6 is independent on $\underline{h}$ and $\{\gamma_k\}$ and has a good convergence property, which is the next lemma.

Lemma 7. *Suppose that the sequences $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are multiplicative. Then V_n converges weakly to $P_{\underline{\zeta}}$ as $n \to \infty$.*

Proof. In [17], the weak convergence for

$$\hat{P}_T(A) = \frac{1}{T}\operatorname{meas}\bigl\{\tau \in [0, T]\colon\ \underline{\zeta}(s + \mathrm{i}\tau; \underline{\mathfrak{a}}) \in A\bigr\}, \quad A \in \mathcal{B}\bigl(H^r(D)\bigr),$$

was considered, and it was obtained its weak convergence to $P_{\underline{\zeta}}$ as $T \to \infty$, and that V_n also converges weakly to $P_{\underline{\zeta}}$ as $n \to \infty$. In other words, V_n and $\hat{P}_T$ have the same limit measure $P_{\underline{\zeta}}$. □

In view of Lemma 7, to prove Theorem 3, it suffices to show that P_N, as $N \to \infty$, and V_n, as $n \to \infty$, have a common limit measure. For this, a certain closeness of $\underline{\zeta}(s; \underline{\mathfrak{a}})$ and $\underline{\zeta}_n(s; \underline{\mathfrak{a}})$ is needed.

There exists a sequence $\{K_l\colon\ l \in \mathbb{N}\} \subset D$ of compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l. Then, putting, for $g_1, g_2 \in H(D)$,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

we have a metric in $H(D)$ inducing its topology of uniform convergence on compacta. Hence,

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} \rho(g_{1j}, g_{2j}),$$
$$\underline{g}_1 = (g_{11}, \dots, g_{1r}),\ \underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D),$$

is a metric in $H^r(D)$ inducing its product topology. Note that, in the proof of the next lemma, the multiplicativity of the sequences $\mathfrak{a}_j$, $j = 1, \dots, r$, is not used.

Lemma 8. *Suppose that estimate* (1) *is true. Then, for every positive* $h_1, \dots, h_r$ *and* $\mathfrak{a}_1, \dots, \mathfrak{a}_r$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \underline{\rho}\big(\underline{\zeta}(s + \mathrm{i}\underline{h}\gamma_k; \underline{\mathfrak{a}}), \underline{\zeta}_n(s + \mathrm{i}\underline{h}\gamma_k; \underline{\mathfrak{a}})\big) = 0. \tag{4}$$

Proof. By the definitions of the metrics $\underline{\rho}$ and ρ, it is sufficient to show that, for every compact set $K \subset D$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s \in K} \big|\zeta(s + \mathrm{i}h_j\gamma_k; \mathfrak{a}_j) - \zeta_n(s + \mathrm{i}h_j\gamma_k; \mathfrak{a}_j)\big| = 0, \tag{5}$$

$j = 1, \dots, r$. The equality of type (5) was already used in [3], therefore, only for fullness, we give remarks on its proof.

Thus, let $h > 0$ and $\mathfrak{a}$ be arbitrary. We consider $\zeta(s + \mathrm{i}h\gamma_k; \mathfrak{a})$ and $\zeta_n(s + \mathrm{i}h\gamma_k; \mathfrak{a})$. Let θ be as in the definition of $v_n(m)$. Then the representation

$$\zeta_n(s; \mathfrak{a}) = \frac{1}{2\pi \mathrm{i}} \int_{\theta - \mathrm{i}\infty}^{\theta + \mathrm{i}\infty} \zeta(s + z; \mathfrak{a}) l_n(z) \frac{\mathrm{d}z}{z},$$

where

$$l_n(z) = \frac{z}{\theta} \Gamma\left(\frac{z}{\theta}\right) n^z,$$

is valid. Hence, for $\theta_1 < 0$,

$$\zeta_n(s; \mathfrak{a}) - \zeta(s; \mathfrak{a}) = \frac{1}{2\pi \mathrm{i}} \int_{-\theta_1 - \mathrm{i}\infty}^{-\theta_1 + \mathrm{i}\infty} \zeta(s + z; \mathfrak{a}) l_n(z) \frac{\mathrm{d}z}{z} + R_n(s; \mathfrak{a}), \tag{6}$$

where

$$R_n(s;\mathfrak{a}) = \frac{al_n(1-s)}{1-s},$$

and a is the residue of $\zeta(s;\mathfrak{a})$ at the point $s=1$. Let $K \subset D$ be an arbitrary compact set, and $\varepsilon > 0$ be such that $1/2 + 2\varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ for $s \in K$. Then, in view of (6), for $s = \sigma + \mathrm{i}v \in K$,

$$\left|\zeta_n(s;\mathfrak{a}) - \zeta(s;\mathfrak{a})\right| \ll \int\limits_{-\infty}^{\infty} \left|\zeta(s-\theta_1+\mathrm{i}t;\mathfrak{a})\right| \frac{|l_n(-\theta_1+\mathrm{i}t)|}{|-\theta_1+\mathrm{i}t|}\,\mathrm{d}t + \left|R_n(s;\mathfrak{a})\right|.$$

Hence, taking t in place of $t+v$ and $\theta_1 = \sigma - \varepsilon - 1/2$, we have

$$\frac{1}{N}\sum_{k=1}^{N} \sup_{s\in K} \left|\zeta(s+\mathrm{i}h\gamma_k;\mathfrak{a}) - \zeta_n(s+\mathrm{i}h\gamma_k;\mathfrak{a})\right| \ll I + Z, \tag{7}$$

where

$$I = \int\limits_{-\infty}^{\infty} \left(\frac{1}{N}\sum_{k=1}^{N} \left|\zeta\left(\frac{1}{2}+\varepsilon+\mathrm{i}h\gamma_k+\mathrm{i}t;\mathfrak{a}\right)\right|\right) \sup_{s\in K} \left|\frac{l_n(1/2+\varepsilon-s+\mathrm{i}t)}{1/2+\varepsilon-s+\mathrm{i}t}\right| \mathrm{d}t$$

and

$$Z = \frac{1}{N}\sum_{k=1}^{N} \sup_{s\in K} \left|R_n(s+\mathrm{i}h\gamma_k;\mathfrak{a})\right|.$$

Estimate (1) is applied for estimation of the first factor of the integrated function in the integral I. It is well known that, for $\tau \in \mathbb{R}$,

$$\int\limits_{0}^{T} \left|\zeta\left(\frac{1}{2}+\varepsilon+\mathrm{i}\tau+\mathrm{i}t;\mathfrak{a}\right)\right|^2 \mathrm{d}t \ll_\varepsilon T\big(1+|\tau|\big). \tag{8}$$

The same estimate is also true for the derivative of $\zeta(s;\mathfrak{a})$. Let $\delta = ch(\log\gamma_N)^{-1}$ and

$$N_\delta(h\gamma_k) = \sum_{\substack{\gamma_k,\gamma_l \leqslant \gamma_N \\ |\gamma_l - \gamma_k| < \delta}} 1.$$

Then, in view of (1) and Lemma 3,

$$\sum_{k=1}^{N} N_\delta(h\gamma_k) = \sum_{\substack{\gamma_k,\gamma_l \leqslant \gamma_N \\ |\gamma_k - \gamma_l| < c(\log\gamma_N)^{-1}}}\sum 1 \ll \gamma_N \log\gamma_N \ll N.$$

This, (6) and an application of the Gallagher lemma connecting discrete and continuous mean squares for some function, see Lemma 1.4 of [27], give

$$
\begin{aligned}
&\sum_{k=1}^{N}\left|\zeta\left(\frac{1}{2}+\varepsilon+\mathrm{i}h\gamma_k+\mathrm{i}t;\mathfrak{a}\right)\right| \\
&\quad\leqslant\left(\sum_{k=1}^{N}N_\delta(h\gamma_k)\sum_{k=1}^{N}N_\delta^{-1}(h\gamma_k)\left|\zeta\left(\frac{1}{2}+\varepsilon+\mathrm{i}h\gamma_k+\mathrm{i}t;\mathfrak{a}\right)\right|^2\right)^{1/2} \\
&\quad\ll N^{1/2}\left(\frac{1}{\delta}\int_{h\gamma_1}^{h\gamma_N}\left|\zeta\left(\frac{1}{2}+\varepsilon+\mathrm{i}\tau+\mathrm{i}t;\mathfrak{a}\right)\right|^2\mathrm{d}\tau\right. \\
&\qquad\left.+\left(\int_{h\gamma_1}^{h\gamma_N}\left|\zeta\left(\frac{1}{2}+\varepsilon+\mathrm{i}\tau+\mathrm{i}t;\mathfrak{a}\right)\right|^2\mathrm{d}\tau\int_{h\gamma_1}^{h\gamma_N}\left|\zeta'\left(\frac{1}{2}+\varepsilon+\mathrm{i}\tau+\mathrm{i}t;\mathfrak{a}\right)\right|^2\mathrm{d}\tau\right)^{1/2}\right)^{1/2} \\
&\quad\ll_{\varepsilon,h} N\bigl(1+|t|\bigr).
\end{aligned}
$$

Therefore, the classical estimate for the gamma-function and the definition of $l_n(s)$ show that

$$
I \ll_{\varepsilon,h,K} n^{-\varepsilon} \quad \text{and} \quad Z \ll_{h,K} n^{1/2-2\varepsilon}\,\frac{\log N}{N}.
$$

This, together with (7), proves (5), thus (4). □

Proof of Theorem 3. We will use the random element language. Denote by $\underline{X}_n=\underline{X}_n(s)$ the $H^r(D)$-valued random element having the distribution V_n, where V_n is the limit measure in Lemma 6. Then, by Lemma 7,

$$
\underline{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P_{\underline{\zeta}}, \tag{9}
$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution. Now, let the random variable η_N be defined on a certain probability space with a measure μ, and

$$
\mu\{\eta_N=\gamma_k\}=\frac{1}{N}, \quad k=1,\dots,N.
$$

Define the $H^r(D)$-valued random element

$$
\underline{X}_{N,n}=\underline{X}_{N,n}(s)=\underline{\zeta}_n(s+\mathrm{i}\underline{h}\eta_N;\underline{\mathfrak{a}}).
$$

Then, in virtue of Lemma 7,

$$
\underline{X}_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} \underline{X}_n. \tag{10}
$$

Let

$$
\underline{Y}_N=\underline{Y}_N(s)=\underline{\zeta}(s+\mathrm{i}\underline{h}\eta_N;\underline{\mathfrak{a}}).
$$

Then Lemma 8 implies that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu\bigl\{\underline{\rho}\bigl(\underline{Y}_n(s), \underline{X}_{N,n}(s)\bigr) \geqslant \varepsilon\bigr\} \leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N\varepsilon} \sum_{k=1}^{N} \underline{\rho}\bigl(\underline{\zeta}(s + \mathrm{i}\underline{h}\gamma_k; \underline{\mathfrak{a}}), \underline{\zeta}_n(s + \mathrm{i}\underline{h}\gamma_k; \underline{\mathfrak{a}})\bigr) = 0.$$

Therefore, this, (9), (10) and Theorem 4.2 of [4] show that $\underline{Y}_N \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{\zeta}}$, and the theorem is proved. □

4 Proof of Theorem 2

We start with the explicit form of the support of the measure $P_{\underline{\zeta}}$. Recall that the support of a probability measure P is a minimal closed set S_P such that $P(S_P) = 1$.

Let $S = \{g \in H(D)\colon\ g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Lemma 9. *The support of the measure $P_{\underline{\zeta}}$ is the set S^r.*

Proof. The space $H^r(D)$ is separable. Therefore [4],

$$\mathcal{B}\bigl(H^r(D)\bigr) = \underbrace{\mathcal{B}\bigl(H(D)\bigr) \times \cdots \times \mathcal{B}\bigl(H(D)\bigr)}_{r}.$$

From this it follows that it suffices to consider the measure $P_{\underline{\zeta}}$ on the rectangular sets

$$A = A_1 \times \cdots \times A_r, \quad A_1, \ldots, A_r \in \mathcal{B}(H(D)).$$

Denote by m_{jH} the Haar measure on Ω_j, $j = 1, \ldots, r$. Then the Haar measure m_H^r is the product of the measures $m_{1H}, \ldots, m_{rH}$. These remarks imply the equality

$$\begin{aligned} P_{\underline{\zeta}}(A) &= m_H^r\bigl\{\omega \in \Omega^r\colon\ \underline{\zeta}(s, \omega; \underline{\mathfrak{a}}) \in A\bigr\} \\ &= m_{1H}\bigl\{\omega_1 \in \Omega_1\colon\ \zeta(s, \omega_1; \mathfrak{a}_1) \in A_1\bigr\} \\ &\quad \cdots m_{rH}\bigl\{\omega_r \in \Omega_r\colon\ \zeta(s, \omega_r; \mathfrak{a}_r) \in A_r\bigr\}. \end{aligned} \tag{11}$$

It is known [19] that the support of

$$P_{\underline{\zeta}_j}(A_j) = m_{jH}\bigl\{\omega_j \in \Omega_j\colon\ \zeta(s, \omega_j; \mathfrak{a}_j) \in A_j\bigr\}, \quad j = 1, \ldots r,$$

is the set S. Therefore, (11) and the minimality of the support prove the lemma. □

Proof of Theorem 2. The theorem is corollary of Theorem 3, the Mergelyan theorem on the approximation of analytic functions by polynomials [25], and Lemma 9, and it is standard. By the Mergelyan theorem, there exist polynomials $p_1(s), \ldots, p_r(s)$ such that

$$\sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} \bigl|f_j(s) - \mathrm{e}^{p_j(s)}\bigr| < \frac{\varepsilon}{2}. \tag{12}$$

In view of Lemma 9, the set

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D)\colon \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\}$$

is an open neighbourhood of an element of the support of the measure $P_{\underline{\zeta}}$. Hence,

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{13}$$

Therefore, by Theorem 3 and the equivalent of weak convergence of probability measures in terms of open sets,

$$\liminf_{N \to \infty} P_N(G_\varepsilon) \geqslant P_{\underline{\zeta}}(G_\varepsilon) > 0.$$

This, the definitions of P_N and G_ε, together with inequality (12), prove the first part of the theorem.

For the proof of the second part of the theorem, we define one more set

$$\hat{G}_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D)\colon \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - f_j(s) \right| < \varepsilon \right\}.$$

Then $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ for all but at most countably many $\varepsilon > 0$, moreover, in view of (12), the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$ is valid. Therefore, Theorem 3, the equivalent of weak convergence of probability measures in terms of continuity sets and (13) lead the inequality

$$\lim_{N \to \infty} P_N(\hat{G}_\varepsilon) = P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. This, the definitions of P_N and $\hat{G}_\varepsilon$ prove the second part of the theorem. □

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